Proceedings of the Tenth Asian Congress of Fluid Mechanics
17-21 May 2004, Peradeniya, Sri Lanka

THE THREE VORTEX PROBLEM AND ITS RAMIFICATIONS

H. Aref
Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University

ABSTRACT: A survey of work on the problem of three interacting point vortices, spanning some 125 years, starting with the 1877 thesis of Gröbli, is reviewed. The pivotal role of three-vortex dynamics in several contexts is highlighted.

1. GENERAL OVERVIEW OF THE LECTURE

One may well question the rationale behind giving a lecture at a state-of-the-art meeting on fluid mechanics in 2004 on such an apparently narrow and classical subject as the motion of three interacting point vortices. The reasons for doing so are set out here.

First, in order to discuss this problem it is necessary to bring together aspects of vortex dynamics and the theory of dynamical systems, and to develop the point vortex equations. This in itself provides a sweep through some not inconsequential fluid mechanics.

Second, the problem of three interacting point vortices is quite rich, a lot richer, for example, than its two-body Kepler problem "counterpart" in the mechanics of gravitating mass points. The mathematical exploration of the three-vortex problem brings out a large number of interesting ideas and constructs, many of which go well beyond the confines of the problem itself: canonical transformations, trilinear coordinates [1, 23], projective geometry [4], and elliptic functions [22] are just some of the mathematical tools that one has the opportunity to see "in action" while solving the three-vortex problem in its various settings.

Third, much of the research literature on few-vortex problems is not well known. Even among those familiar with the point vortex equations – and most fluid mechanicians are – the classic work by Gröbli in the late 19th century on the reduction of the three-vortex problem to quadratures is not widely appreciated. And among those who at a formal level understand that the three-vortex problem must be integrable, the details of this problem, i.e., an understanding of the different types of motion that arise as the vortex strengths are varied, is usually lacking. The answer to the question: "What happens in the three-vortex problem?" is long and relatively complex.

The dynamical system defined by three point vortices is integrable on the unbounded plane and in a periodic parallelogram (where periodicity guarantees that the circulations of the three vortices sum to zero). It is not integrable, in general, if the vortices are enclosed in a circular domain, although the two-vortex problem is. For general domains only the one-vortex problem is integrable. Physical many-body systems for which the three-body problem is integrable (without the N-body problem for all larger N being integrable) are rare, so the three-vortex problem is intrinsically interesting for this reason alone. In the two-vortex problem on the unbounded plane the distance between the two vortices is a constant of the motion. One has to go to three vortices in order to achieve relative motion of the vortices and thus temporal changes in the "scales of motion" defined by the inter-vortex distances. This identifies three-vortex motion as the simplest "generator" of the kind of configurations observed in two-dimensional turbulence, a model problem of considerable importance to our understanding of atmospheric and oceanographic flows. For all these reasons some time spent studying the three-vortex problem is not without its rewards.

Fourth, as my understanding of this problem has progressed, it has become increasingly clear that a substantial number of flow phenomena can be explained, or at least partially understood, by appealing to 2D few-vortex motions. There has been a feedback loop at work here, of course: Some experiments have been done precisely because point vortex results suggested that interesting behavior would arise. On the other hand, as we learn more about what few-vortex systems can do, various experimental results will yield to interpretation in these terms.

The history of few-vortex dynamics in general, and the three-vortex problem in particular, displays an intriguing pattern of discovery and re-discovery. Key results were obtained and then forgotten only to be re-obtained decades later. The interested researcher must, therefore, piece together an account from papers in diverse journals in several languages written over more than a century. The cast of contributors includes several of the great fluid mechanicians and applied mathematicians whose work can be found with relative ease in any modern scientific library: Helmholtz [14, 24], Kirchhoff, Kelvin, Poincaré [19] and Zhukovskii. Two key contributions, however, are the thesis by an almost entirely forgotten Swiss mathematician, W. Gröbli [12], who flourished briefly in research during the period 1875-1880, and a paper by the distinguished applied mathematician J. L. Synge [23], published in the first volume of *Canadian Journal of Mathematics.* Both works have remained unappreciated and largely forgotten until recently.

In this context it is interesting to look up various well known texts to see what the casual reader might derive is known about the N-vortex problem for $N \geq 3$. Among the books that treat point vortex dynamics (and not all do) essentially all give the solution of the two-vortex problem. Many do not even mention the three-vortex problem, presumably on the assumption that it is as complicated as the (generally nonintegrable) three-body problem of celestial mechanics. The singularly influential text by Batchelor [9] contains the following: "When $N = 3$, the details of the motion are not evident, but the ... invariants [of the problem] suggest that all three vortices remain within a distance ... [set by the initial conditions] from the centre of vorticity (except in a case in which the sum of two of [the circulations] is close to zero) and that the distance between any two vortices can never be much less than the smallest distance between any pair of vortices initially." (This turns out to be false in the particular case of what is known as "vortex collapse" [1, 17].) In the classic text by Lamb [16] one finds a footnote reference to the work of Gröbli, with the understated comment that the paper "contain[s] other interesting examples of rectilinear vortex-systems."

Modern interest in the three-vortex problem owes much to a 1975 paper by E. A. Novikov [18]. In essence, Novikov independently re-discovered the solution for the motion of three identical vortices that Gröbli had found a century before. In 1979 while generalizing Novikov's results, and being also unaware of Gröbli's and Synge's 1949 work, I rediscovered much of what they had done [1]. The reason both Novikov's work and my generalization were published in reputable journals was, of course, that the earlier contributions had been entirely forgotten by the fluid mechanics community, including by the referees of our papers. None of the experts in vortex dynamics knew that the problem had already been solved. Hence, both as a scientific problem and as a curious chapter in the history of fluid mechanics the three-vortex problem has held a particular fascination.

Sometime in the Fall of 1985 I received a telephone call from Nicholas Rott, a distinguished Swiss aerodynamicist retired from the Eidgenössische Technishe Hochschule (ETH) in Zürich working at Stanford University. Rott had read my review article [2] in which there is a brief mention of the history of the three-vortex problem and references to the works by Gröbli and Synge. He called to tell me that when he was a student at the ETH, his professor Jakob Ackeret, one of the "grand old men" of aerodynamics, had regularly included in his lectures a description of the three-vortex problem and had shown illustrations from the work by Gröbli. This contact and mutual interest led to discussions on how one might make Gröbli's work more accessible and rescue it from obscurity. A number of obvious issues were considered: Would it be appropriate to publish a full translation of Gröbli's thesis? What had subsequently happened to Gröbli whose dissertation work had clearly been so very well received? Had he done other work in fluid mechanics or mathematics? And so on.

It was ultimately agreed that publication of a translation of a century old dissertation was probably not of sufficient interest to modern day readers – some indication of more recent developments would be required. A detailed historical study was probably not appropriate either, nor were we the right individuals to undertake it. In the end, with the help of some successful detective work by Prof. Hans Thomann, Rott's successor at the ETH, a historical appreciation of Gröbli's work was prepared [7].

Rott went on to do an intriguing analysis of the case of three vortices with zero net circulation [3, 20], which was later generalized substantially to the case of three vortices in a domain with periodic boundary conditions [8, 22].

The lecture attempts, on one hand, to give Gröbli's contribution the recognition and accessibility it deserves and, on the other, to provide the necessary up-to-date account of what has happened since, allowing the modern reader to see the three-vortex problem, and some aspects of what we know about the N-vortex problem for $N \geq 4$ [6, 10, 11, 21], in a richer, more complete context. We encounter quite a spectrum of phenomena, including stationary patterns of vortices in superfluids and electron plasmas [5], chaotic motions of a few vortices, aspects of the dynamics of wakes [15], the famous "vortex tripole" [13], and the ubiquitous problem of two-dimensional turbulence.

ACKNOWLEDGMENTS

My understanding of the three-vortex problem and its role has evolved over the years through collaborations with (in alphabetical order) Phil Boyland, Morten Brøns, Bruno Eckhardt, Yoshi Kimura, Slava Meleshko, Paul Newton, Neil Pomphrey, Nicholas Rott, Mark Stremler, Dmitri Vainchtein and Irek Zawadzki. To all of them I extend my thanks.

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