

DYNAMICS OF SHEARED GRANULAR FLUID¹

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ABSTRACT: *The dynamics of sheared granular fluid is briefly reviewed, focusing on instabilities, patterns and bifurcations in plane shear flow. It is shown that a universal criterion holds for the onset of the shear-banding instability (for perturbations having no variation along the stream-wise direction), that lead to shear-band formation along the gradient direction. The same shear-banding criterion appears to hold in other complex fluids as well as in the singular limit of atomistic fluids (i.e. elastic hard-spheres). A weakly nonlinear analysis of the shear-banding instability unveils that the lower branch of the neutral stability curve, that corresponds to dilute flows, is sub-critically unstable. In the presence of gravity, the origin of such shear-banding transition is shown to be tied to the spontaneous symmetry-breaking shear-banding instabilities of the gravity-free uniform shear flow, resulting in universal unfolding of pitchfork bifurcations in gravity-modulated plane shear flow.*

1 INTRODUCTION

The collective motion of a large number of macroscopic solid particles is called “granular” flow, and an example of such flow is the gravity-driven motion of particles down an inclined plane. In typical “dry” granular flows, the effect of interstitial fluid is neglected and the interactions between grains are *dissipative*, which, in turn, leads to a wealth of interesting behaviour^[1–5]. The granular fluid differs from its atomistic/molecular counterpart in that the collisions between macroscopic particles are inherently *inelastic*. This implies that if there is no external supply of energy into the system, the fluctuation kinetic energy (i.e. *granular temperature*) would eventually decay. Thus, to maintain a granular flow in its fluidized state, energy must be supplied to the system, for example, by shearing or shaking. Unlike Brownian particles, the potential energy of a grain is much larger than its thermal energy, and hence the granular matter is a prime example of an “athermal” system. The behaviour of granular materials is of immense importance in many industrial and geological processes; most agricultural and pharmaceutical products are in granular form. Granular materials are encountered in everyday life: sand, gravel, sugar, salt, cereals and powders. Despite their practical importance and non-trivial dynamics, the current understanding of granular flows still remains at its infancy.

Instability-induced patterns have been extensively studied in classical fluid mechanics over more than a century. Pattern formation in rapid granular flows (plane Couette flow, Poiseuille flow, vibrated bed, etc.) has received considerable attention during the last few years. Recent observations of cluster-formation, density waves and stress-fluctuations in particle dynamics simulations of granular flows have motivated analyses of their stability. For experiments down an inclined plane, many interesting patterns in the form of roll waves, fingering instability and longitudinal vortices have recently been reported^[2]. From the theoretical viewpoint, an immediate important question is whether one could explain such pattern-formations from a minimal set of continuum equations. In the *rapid-shear* regime^[1], an analogy between the collisional granular fluid with a dense molecular gas has led to the development of the kinetic-theory-based constitutive models^[3]. These hydrodynamic models, typically truncated at the Navier-Stokes’ (NS) order, have been widely used to gain insight into the macroscopic behaviour of various physical phenomena involved in dry granular flows.

The plane Couette flow is a prototype model problem to study the rheology^[1,3] and dynamics^[1,2,6–11] of granular materials. In the rapid shear regime, the linear stability analyses showed that the plane Couette flow admits different

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types of *stationary* and *traveling* wave instabilities, leading to different types of patterns^[6,7]. Among all instabilities, one interesting instability is the ‘shear-banding’ instability for which the homogeneous shear flow breaks into alternating dense and dilute regions of low and high shear rates, respectively, along the gradient direction. This is dubbed *shear-banding* instability since the “nonlinear” saturation of this instability is reminiscent of shear-banding solutions in typical shear-cell experiments^[4]: *when a dense granular material is sheared the shearing is confined within a few particle-layers (i.e., a shear-band) and the rest of the material remains unsheared*, leading to the two-phase flows of dense and dilute regions.

After presenting some simulation results in Sec. 2, I review previous linear and nonlinear stability results^[6–11] of plane Couette flow in Secs. 3 and 4. In particular, I will discuss the origin of the shear-banding instability and its universality from the viewpoint of hydrodynamic stability.

2 PATTERNS IN PLANE COUETTE FLOW: SIMULATION

Consider a mono-disperse system of smooth inelastic hard-disks (of diameter d and material density ρ_p) in a square-box of size \tilde{H} under uniform shear flow. The system is periodic in \tilde{x} -direction (i.e. a particle crossing the left/right boundary re-enters the system through the opposite boundary at the same vertical position with unchanged velocities), and the standard Lees-Edwards boundary condition is used in the \tilde{y} -direction to impose a shear-field across \tilde{y} -direction. This mimics a shear flow which is driven by two oppositely moving plates. In a typical simulation, the disks are initially placed randomly in the computational box, and the initial velocity field is composed of the uniform shear and a small Gaussian random part. An event-driven algorithm is then used to update the system in time. When two particles collide, they lose energy due to the inelastic nature of collisions which is characterized by the coefficient of restitution ($0 \leq e \leq 1$). For non-dimensionalization, we use \tilde{H} , $\tilde{\gamma}^{-1}$ and $\tilde{\gamma}\tilde{H}$ as the reference scales for length, time and velocity, respectively, with $\tilde{\gamma}$ being the imposed shear rate. Thus the computational box spans the range of $[-0.5, 0.5]$ in both the stream-wise and transverse directions; the top image-box moves with a velocity 0.5 and the bottom with -0.5 . Depending on control parameters, different types of patterns emerge in plane shear flow, and below I discuss two snapshots of shear-banding pattern.

Figure 1(a) shows an interesting pattern, with the particles forming a dense *plug* around the center-line. This corresponds to a very dilute system, with the mean volume fraction of particles being $\phi = 0.05$. For this case, the total number of particles is $N = 15000$ and the restitution coefficient is $e = 0.9$. Starting from an unsheared initial condition of random particle-configurations, this snapshot represent a steady state of the system. The corresponding coarse-grained density profile is shown in Fig. 1(b)—thus, a homogeneous system splits into two parts, a dense, cold crystalline area and a dilute, hot, fluid area. The shear-rate is not uniform along the gradient direction y : low/high shear-rate in the dense/dilute part, respectively, leading to *shear-localization* or *shear-band* formation along the gradient direction. This is reminiscent of *shear-band* formation in many other complex fluids^[12].

A similar shear-banded solution^[8] is shown in Fig. 2 for a moderately dense system $\phi = 0.3$. From an unsheared initial condition of random particle-configurations, this system has evolved in time to form two plugs near the walls, with a very dilute region of particles in the bulk of the shear-cell. With the same parameter combinations, we have checked that this system also evolves to yield a single central-plug, located symmetrically around the x -axis. Both of these solutions satisfy the underlying symmetries of the shear flow and hence are permissible.

In Section 4, I will comment on the predicted density profiles under similar conditions that arise out of bifurcations from the “homogeneous” shear flow. The related hydrodynamic model is described next.

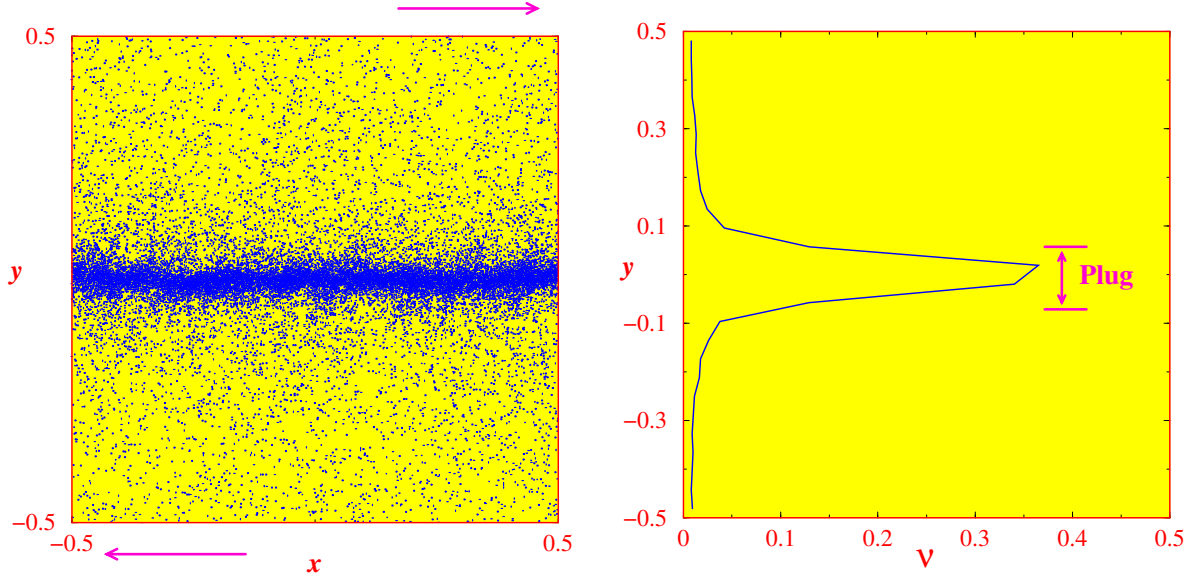


Figure 1: (a) Spontaneous shear-banding in a low-density sheared granular fluid: $\phi = 0.05$, $N = 15000$ and $e = 0.9$; (b) coarse-grained density profile.

3 HYDRODYNAMIC MODEL AND STABILITY

Restricting to the Navier-Stokes-level description of a granular fluid, we write down the balance equations for mass, momentum and granular energy^[3]:

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \varrho = -\varrho \nabla \cdot \mathbf{u} \quad (1)$$

$$\varrho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = \varrho \mathbf{g} - \nabla \cdot \mathbf{P} \quad (2)$$

$$\frac{dim}{2} \varrho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) T = -\nabla \cdot \mathbf{q} - \mathbf{P} : \nabla \mathbf{u} - \mathcal{D}. \quad (3)$$

Here $\varrho = mn = \rho_p \phi$ is the mass-density, m the particle mass, n the number density, ρ_p the material density and ϕ the volume fraction of particles; \mathbf{u} is the coarse-grained velocity-field and T is the granular temperature of the fluid; \mathbf{g} is the gravitational acceleration and dim is the dimensionality of the problem ($= 2/3$ in two/three dimensions, respectively). Note that the granular temperature, $T = \langle C^2/3 \rangle$, is defined^[3] as the mean-square fluctuation velocity, with $\mathbf{C} = (\mathbf{c} - \mathbf{u})$ being the peculiar velocity of particles and \mathbf{c} the instantaneous particle velocity. The flux terms are the stress tensor, \mathbf{P} , and the granular heat flux, \mathbf{q} ; \mathcal{D} is the rate of dissipation of granular energy per unit volume. The constitutive relations for these flux terms needed which are detailed below.

3.1 Constitutive Relations

The standard Newtonian form of the stress tensor and the Fourier law of heat flux are:

$$\mathbf{P} = (p - \zeta \nabla \cdot \mathbf{u}) \mathbf{I} - 2\mu \mathbf{S} \quad \text{and} \quad \mathbf{q} = -\kappa \nabla T \quad (4)$$

where \mathbf{I} is the identity tensor and \mathbf{S} is the deviator of the deformation rate tensor. Here p , μ , ζ and κ are pressure, shear viscosity, bulk viscosity and thermal conductivity of the granular fluid, respectively.

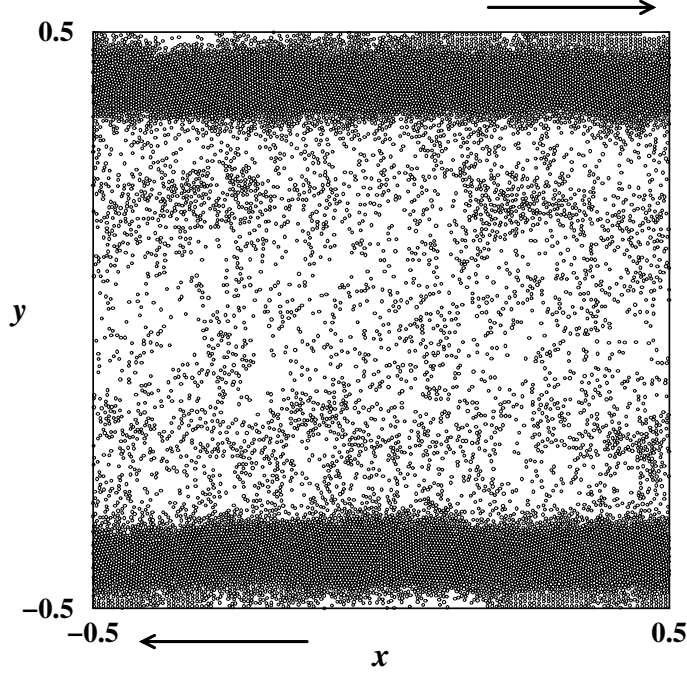


Figure 2: Same as Fig. 1(a) for a moderate density $\phi = 0.3$, with $e = 0.8$ and $N = 15000$. (Adapted from JFM, vol. 523, p. 277)

Focusing on the nearly elastic limit ($e \sim 1$) of an inelastic hard-sphere fluid, the constitutive expressions for p , μ , ζ , κ and \mathcal{D} are given by^[3]

$$\begin{aligned} p(\phi, T) &= \rho_p f_1(\phi) T, & \mu(\phi, T) &= \rho_p d f_2(\phi) \sqrt{T} \\ \zeta(\phi, T) &= \rho_p d f_3(\phi) \sqrt{T}, & \kappa(\phi, T) &= \rho_p d f_4(\phi) \sqrt{T} \\ \mathcal{D}(\phi, T) &= \frac{\rho_p}{d} f_5(\phi, e) T^{3/2} \end{aligned} \quad (5)$$

where f_1 – f_5 are dimensionless functions of the particle volume fraction. It may be noted that the dissipation of energy, \mathcal{D} , is identically zero for elastic ($e = 1$) hard-spheres.

3.2 Plane Couette Flow: Base Flow

We will study the stability of the plane Couette flow of a granular fluid bounded by two oppositely-moving walls as discussed in Sec. 2. For the steady ($\partial/\partial t = 0$), fully developed ($\partial/\partial y = 0$) shear flow, the continuity equation is identically satisfied, and the momentum and energy balances take the following forms

$$\left. \begin{aligned} \frac{d}{dy} \left(\mu \frac{du}{dy} \right) &= 0, & \frac{dp}{dy} + \frac{\phi H^3}{Fr^2} &= 0, \\ H^{-2} \frac{d}{dy} \left(\kappa \frac{dT}{dy} \right) + \mu \left(\frac{du}{dy} \right)^2 - \mathcal{D} &= 0. \end{aligned} \right\} \quad (6)$$

There are four control parameters: the coefficient of restitution e , the mean density $\phi_{av} = \phi$, the Couette gap (i.e. non-dimensional wall separation) $H = \tilde{H}/d$, and the Froude number $Fr = \tilde{U}/\sqrt{gd}$. The *gravity-free* case can be obtained by considering the infinite shear-rate limit, i.e. $Fr = \infty$.

For the simplest case with *no-slip* and *zero* energy-flux boundary conditions, it can be verified that the gravity-free case admits a uniform shear solution with constant solid fraction and granular energy:

$$\phi(y) = \text{const.}, \quad u(y) = y, \quad T(y) = f_2(\phi)/f_5(\phi, e). \quad (7)$$

Before proceeding further, we note that the equations (6) with $Fr = \infty$ admit the following symmetry:

$$\phi(y) = \phi(-y), \quad u(y) = -u(-y), \quad T(y) = T(-y). \quad (8)$$

The effects of boundary conditions on instabilities and related bifurcated solutions have been discussed elsewhere [7].

3.3 Stability Analysis

To study linear stability of the steady, fully developed plane Couette flow [Eqn. (7)], the base flow is perturbed by infinitesimal disturbances, and their time evolution is studied by linearizing the governing equations about the base state. Since the linearized disturbance equations and boundary conditions do not depend on t explicitly, the normal-mode ansatz can be used (we restrict to two-dimensions):

$$[\phi', u', v', T'](x, y, t) = [\hat{\phi}(y), \hat{u}(y), \hat{v}(y), \hat{T}(y)] e^{ik_x x + \omega t}, \quad (9)$$

where quantities with hats are complex amplitude functions of y , k_x is the stream-wise wavenumber and ω the disturbance frequency. For temporal stability, k_x is assumed to be real and ω is complex. The rate of growth or decay of disturbances is determined by ω_r , the real part of ω , and the imaginary part is the frequency. The flow is stable, neutrally stable, or unstable accordingly as ω_r is negative, equal to zero, or positive, respectively.

Using (9) we obtain a set of linear ordinary differential equations for the disturbance amplitudes $\hat{X} = (\hat{\phi}, \hat{u}, \hat{v}, \hat{T})$:

$$\mathcal{L}(\cdot)\hat{X} = \omega\hat{X}, \quad \text{with} \quad \mathcal{B}_{\pm}(\cdot)\hat{X} = 0 \quad \text{at} \quad y = \pm 1/2, \quad (10)$$

which constitutes an eigenvalue problem with ω as its eigenvalue. The explicit forms of $\mathcal{L}(\cdot)$ and the boundary operators $\mathcal{B}_{\pm}(\cdot)$ can be found elsewhere^[7,8]. A staggered grid spectral collocation scheme^[7] is used to discretize the stability equations (10) in the gradient direction. The discretized stability equations along with the boundary conditions are formulated as a matrix eigenvalue problem: $\mathbf{A}\Phi = \omega\mathbf{B}\Phi$, where ω is the eigenvalue and Φ is the discrete analogue of the eigenfunction.

4 DISCUSSION: SHEAR-BAND INSTABILITY AND UNIVERSALITY

Previous works^[5–11] have unveiled that the uniform granular shear flow is unstable to various kinds of disturbances, leading to both *stationary* and *traveling* wave patterns. Here we consider only a special kind of *stationary* instability that arises due to purely stream-wise disturbances ($k_x = 0$), i.e. the disturbance patterns do not vary with x ; we call it *shear-banding instability* since its nonlinear saturation leads to shear-banding-type patterns of alternating bands of dilute and dense regions in the gradient direction. For such disturbances, there is a minimum value of solid fraction ($\phi \sim \phi_c$) above which the flow is unstable if the Couette gap satisfies the following relation^[7]:

$$H \geq n\pi\psi(\phi, e), \quad (11)$$

where $\psi(\phi, e)$ is a complicated function of density and restitution coefficient, and $n = 1, 2, \dots$ is the mode number (which is related to the eigenfunctions of the linearized stability problem). It is clear that the $n = 1$ mode is the first to become unstable at a critical value of the Couette gap $H = H_c \equiv \pi\psi(\phi, e)$ (for given ϕ and e). Beyond this minimum Couette gap $H > H_c$, the successive higher-order modes ($n = 2, 3, \dots$) take over as the most unstable mode at $H = nH_c$. Treating H as a bifurcation parameter, there is a countably infinite number of pitchfork bifurcations (since the least-stable eigenvalue is real), located at $H = nH_c$ ^[7,8]. From a numerical bifurcation analysis^[8], we obtained nonlinear density profiles that show segregation of particles along the gradient direction, similar to those displayed in Figs. 1 and 2.

A weakly nonlinear analysis^[11] of the shear-banding instability shows that the lower branch of the neutral stability curve, that corresponds to dilute flows, is sub-critically unstable. This explains the viability of the shear-banding state in dilute flows (such as in Fig. 1) for which the linear stability theory predicts *stability* of the homogeneous state.

In the presence of gravity^[8], the origin of such shear-banding transition has been shown to be tied to the spontaneous symmetry-breaking shear-banding instabilities of the gravity-free uniform shear flow. The gravity plays the role of an *imperfection* and all possible forms of imperfect bifurcation scenarios can be realized in the present bifurcation

problem. Thus, the effect of gravity on the granular Couette flow truly belongs to the class of the *universal unfolding* of pitchfork bifurcations^[8].

Using linear stability theory and a numerical bifurcation analysis, our recent work^[9,10] has uncovered that the sheared granular flow evolves toward a state of lower *dynamic friction*. Beyond a critical density, the homogeneous state corresponds to a relatively *higher* dynamic friction, and hence the shear flow jumps into a state of *lower* dynamic friction that corresponds to a shear-banding state of segregated density profile along the gradient direction^[9]. Interestingly, this finding also helps to explain many previous particle dynamics simulation results on the decrease of dynamic friction at very high densities^[3].

Recently, I uncovered an interesting analogy^[10] of the present shear-banding criterion with the shear-banding phenomenon in a variety of other complex fluids (e.g., worm-like micelles, colloidal suspensions, polymeric melt, biphasic liquid system, etc.) under shear^[12]. Such shear-induced banding in complex fluids has been tied to a decrease in viscosity, or, equivalently, a lower viscous dissipation^[12]. This is similar to our universal criterion of a lower “dynamic” friction for the shear-banding state in a sheared granular fluid^[9,10].

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