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Non-normality and Nonlinearity in Thermoacoustic Instabilities

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Abstract

Thermoacoustic instability has traditionally been investigated by linearizing the equations of combustion-acoustic interaction and testing for unstable eigenvalues of the linearized problem. However, it was observed that often the results of linear stability analysis agree poorly with experiments. Nevertheless, linear effects play a central role in combustion instability. The consequence of non-normality in the occurrence of subcritical transition to instability is illustrated in the context of a horizontal Rijke tube. It is shown that the coupled thermoacoustic system is non-normal as well as nonlinear. Non-normality can cause algebraic growth of oscillations for a short time even though all the eigenvectors of the system could be decaying exponentially with time. This feature of non-normality combined with the effect of nonlinearity causes the occurrence of subcritical transition to instability from initial states that have small energy. Measures to quantify transient growth are also discussed. Examples discussed include thermoacoustic instabilities in ducted premixed and diffusion flames and solid rocket motor.

Keywords

Thermoacoustic instability, non-normality, nonlinearity, subcritical bifurcation, triggering

1. Introduction

The occurrence of thermoacoustic instabilities, also known as combustion instabilities has been a plaguing problem in the development of combustors for rockets, jet engines and power generating gas turbines [1]. Predicting and controlling thermoacoustic instability requires an understanding of the interactions between the combustion process and the acoustic wave.

Thermoacoustic instability arises primarily from an interaction of acoustic waves and unsteady heat release. The occurrence of thermoacoustic instability depends upon the phase between heat release rate fluctuations and pressure fluctuations at the flame. According to Rayleigh [2], amplification of the pressure oscillations by the heat addition process will take place if the maximum and minimum of the heat addition occur during the compression and rarefaction phases of the pressure oscillation, respectively. In contrast, the pressure oscillations will be attenuated if the maximum and minimum of the heat addition occur during the rarefaction and compression phases of the pressure oscillation, respectively.

When pulsations start spontaneously, the system is said to be linearly unstable; i.e., the system is unstable with respect to any small amplitude disturbance that may be present in the combustor. This scenario has been successfully investigated for various thermoacoustic systems by using a linear stability analysis that model the system as a network model. In a network model, each element is modeled using a linear transfer function [3, 4]. The stability of the system can then be determined easily by examining the eigenvalues of the system.

It is also possible that a linearly stable combustor (i.e., one that does not pulse spontaneously) could be "triggered" into pulsating operation by the introduction of a finite amplitude disturbance such as might be caused by a spark plug ignition or a small explosion. Such a system will be stable with respect to all disturbances whose amplitudes are below a certain threshold value, but will transition into pulsating operation when the amplitude of the disturbance exceeds this threshold value. Such subcritical bifurcations cannot be explained by using classical linear stability analysis.

Although much work has been done over the last 50 years, it is mostly in the framework of classical linear stability analysis. A comprehensive prediction of the conditions for the onset of instabilities is a difficult task, which is not yet mastered. In particular, predicting the conditions under which finite-amplitude disturbances destabilize a linearly stable system and predicting the limit-cycle amplitude of the instability remain a key challenge, as little is known, even in a qualitative sense, about the key parameters controlling nonlinear flame dynamics, even in simple laminar flames [5].

In order to predict the occurrence of subcritical bifurcations and to calculate the limit cycle amplitude, a nonlinear theory of thermoacoustic oscillations is necessary. The nonlinear response of the oscillatory heat release to velocity fluctuations has been investigated by a number of authors in the context of various burners that show thermoacoustic instabilities. Here, we will review the literature on nonlinear effects in a Rijke tube, which we will be examining in Section 3 in some detail. Nonlinear effects in Rijke tube have drawn considerable attention in the recent years. A Rijke tube is a relatively simple system: it is a duct which is open at both ends and has a heat source (often electrically heated wires) placed near quarter length from the bottom (if located vertically). Heckl [6] studied the nonlinear effects leading to limit cycles, both experimentally and theoretically for the case of a Rijke tube. She showed that the important nonlinear effects are the reduction of the rate of heat transfer when the velocity amplitudes are of the same order as the mean velocity and increased losses at the ends of the tube at very high amplitudes. Hantschk and Vortmeyer [7] showed that the limit-cycle amplitude in a Rijke tube is determined by nonlinearities in the heat flux from the heating element to the flow. Yoon et al. [8] proposed a nonlinear model of a generalized Rijke tube. Their model for the oscillatory heat release rate was not derived from physical principles. They derived both closed form and numerical solutions for the acoustic field by an approximate modal analysis using a two-mode formulation. The two-mode nonlinear model is capable of predicting the growth of oscillations in an initially decaying system. Yoon et al. [8] refers to this as the bootstrapping effect, which they say, characterizes nonlinear velocity sensitive combustion response in rocket motors. However, they neglected nonlinear acoustics in their model. In order to explain the nonlinear effects of a Rijke tube, Matveev [9, 10] constructed a simple theory by using an energy approach. The equilibrium states of the system are found by balancing thermoacoustic energy input and acoustic losses. His work reaffirmed that the nonlinearity of the unsteady heat transfer is a dominant factor in determining the limit-cycle amplitude. Further, Matveev & Culick [11] demonstrated the necessity of accurately modeling the effects of temperature gradient on the mode shapes to obtain accurate results for stability.

Although considerable research has been performed on the nonlinear nature of thermoacoustic oscillations, their non-normal nature is an aspect that has not received any attention until recently. Non-normality can lead to transient growth of oscillations in a system even when the eigenvalues indicate linear stability. However, the transient growth will be followed by an asymptotic decay for the linearized system. There could be situations where the short term growth of fluctuations can lead to significant amplitudes, where nonlinear effects could cause ‘nonlinear driving’, causing the linearly stable system to evolve to a limit cycle.

The role of non-normality in thermoacoustic oscillations has been shown in the context of ducted diffusion flames [12], Rijke tube [13-15], premixed flames [16] and vortex combustors [17] using a Galerkin type analysis. Nicoud et al. [18] has shown that the eigenvectors of a thermoacoustic system are non-orthogonal in the presence of heat release, or, in the presence of general complex impedance boundary conditions. Kedia et al. [19] showed that ignoring the linear coupling of the modes will result in significant changes in the linear and nonlinear system dynamics. Mariappan and Sujith [20] investigated non-normality in the context of pulsed instabilities in solid rocket motors. Kulkarni et al. [21] demonstrated that non-normality and transient growth can lead to the failure of traditional controllers that are designed based on classical linear stability analysis.

Thus there are two distinct aspects of combustion acoustic interactions: 1) non-orthogonality of the eigenmodes of the linearized system and 2) nonlinearities. The objective of this paper is to highlight these aspects, particularly in the context of subcritical bifurcations. The rest of this paper is organized as follows. In Section 2, the characteristics of a non-normal operator and its consequences are discussed in the context of thermoacoustic instabilities. Non-normality and its consequences in subcritical transition to instability are explained in Section 3, using a toy model of a horizontal Rijke tube. Non-normality in ducted premixed flame and diffusion flames are explained in Section 4 and 5 respectively, using simple models drawn from the literature. Section 5 discusses non-normality in the context of pulsed instabilities in solid rocket motors. The outlook for the future is discussed in Section 6.

2. Non-normality and Transient Growth in Thermoacoustic Oscillations

Classical linear stability analysis is a standard tool for studying the stability of oscillations in combustors [1, 22, 23]. This involves examining the evolution of small perturbations by linearizing the dynamical system about the unperturbed state and examining the eigenvalues of the linearized system. If the real parts of all the eigenvalues are negative, then the system is said to be linearly stable. If at least one of the eigenvalues has a positive real part, the system is linearly unstable with an exponential growth in the amplitude of oscillations. In other words, linear stability analysis gives conditions for asymptotic stability. However, if a linearly stable system is non-normal, the small perturbations may exhibit large transient energy growth before their eventual decay.

This property is illustrated in Figure 1a, where x_1 and x_2 represent the directions of the eigenvectors and R vector is a linear combination of the eigenvectors with components r_1 and r_2 along x_1 and x_2 respectively. As illustrated in Figure 1a, a normal system has orthogonal eigenvectors. If the amplitudes of these individual eigenvectors decay, then R decreases monotonically. On the contrary, Figure 1b shows that for a non-normal system, R may increase even when the amplitudes of individual eigenvectors, which are non-orthogonal, decay. During such transient growth, nonlinear effects may get triggered resulting in the growth and eventual saturation of the perturbations, thereby changing the asymptotic stability of the system as illustrated in Figure 2. Reddy and Henningson [24] provide a comprehensive analysis on non-normal transient energy growth in the context of hydrodynamic instabilities in a viscous channel flow. A similar scenario arises in the case of thermoacoustic oscillations as well.

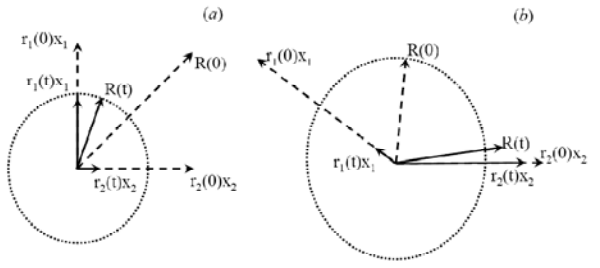

Figure 1: (a) shows decay of a normal system, (b) shows transient growth of a non-normal system. The initial state is $R(0) = r_1(0)x_1 + r_2(0)x_2$, and the final state $R(t) = r_1(t)x_1 + r_2(t)x_2$. The dashed lines denote the vectors at time $t = 0$ and the solid lines denote the vector at some time *t*.

Neglecting the effect of mean flow on wave propagation, the acoustic equations in the presence of a heat source can be written as [25]:

$$\gamma M \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0 \qquad (1)$$

$$\frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = (\gamma - 1)\gamma \frac{L_a}{c_0} \frac{\dot{Q}'}{\rho_0 c_0^2} \qquad (2)$$

The assumptions in deriving these equations and the details of non-dimensionalisation are given in Section 3.1.

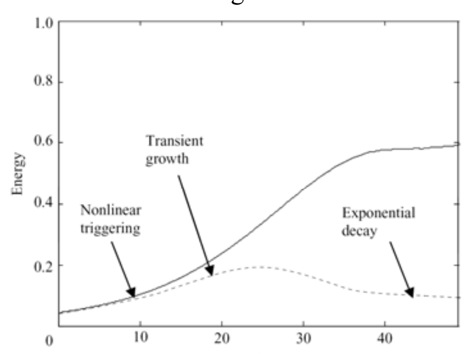

Figure 2: Schematic of linear and nonlinear evolution of acoustic energy obtained from linear and nonlinear simulations $\left(1/2\, p'^2/(\gamma M)^2 + 1/2\, u'^2\right)$; reproduced from reference [14], with permission from Cambridge University Press.

The heat release rate in the above equation is calculated from a model for the heat source. The linearized oscillatory heat release rate, can be written as $\dot{Q}' = R(x, \varepsilon_i)\gamma M u' + S(x, \mu_i) p'$, where *R* and *S* can be treated as a continuous function of *x* (which could even be sharply peaked at the flame location as in the case of a compact heat source), ε_i and μ_i are parameters which affect heat release rate. The heat release rate could have an explicit dependence on time as well. Equations (1) & (2) can be recast in the matrix form as:

$$\frac{\partial}{\partial t}\begin{bmatrix} \gamma M u' \\ p' \end{bmatrix} + \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} - \frac{R L_a(\gamma - 1)}{\rho_0 c_0^3} & -\frac{S L_a \gamma(\gamma - 1)}{\rho_0 c_0^3} \end{bmatrix}\begin{bmatrix} \gamma M u' \\ p' \end{bmatrix} = 0 \qquad (3)$$

The matrix in Eq. (3) is a matrix of operators. The above operator does not commute with its adjoint for non-zero *R* and *S*. (The adjoint of a real matrix is simply the transpose of the matrix. The adjoint of a differential operator is its negative of its transpose.) Therefore, it is clear that thermoacoustic interactions are non-normal. In the absence of heat release fluctuations and for perfect boundary conditions, the matrix is symmetric and hence normal for classical boundary conditions.

Convection plays a dominant role in combustors, and the combustion process. The combustion process is often modelled using advection diffusion equation for the Schvab – Zeldovich variable in the case of diffusion burners and advection-reaction equation in the case of premixed flames. It has be shown that the advection-diffusion operator is non-normal [26, 27]. Subramanian & Sujith [16] has shown that the advection-reaction equation for a premixed flame is non-normal.

As a result of the non-normal behavior, the solutions exhibit large transient growth which could potentially trigger nonlinearities in the system when the amplitudes reach high enough values. Under such circumstances, classical linear stability analysis becomes a poor indicator of the stability of the system [28]. Non-normality and transient growth and their consequences on hydrodynamic stability have been studied in detail in the context of turbulence by Baggett, Driscoll & Trefethen [29]. They explained that in the non-normal evolution, the input and output structures (such as streamwise vortices, streaks etc.) are different and nonlinearity closes the feedback loop by converting some of the output energy into input. In thermoacoustic systems, transient growth resulting from non-normality in combination with nonlinear effects can lead to the growth of the acoustic oscillations over a large number of cycles.

The consequences of non-orthogonality of eigenmodes have been studied in the context of instability of magnetic plasmas by Kerner [30], atmospheric flows by Farrell [31] and transition to turbulence by Baggett, Driscoll & Trefethen [29], Trefethen *et al*. [28] and Gebgart and Grossman [32].

In order to analyze equation (3), the operator has to be reduced to a finite dimensional matrix. In Section 2, the partial differential equations governing the thermoacoustic interaction are reduced to a set of ordinary differential equations using the Galerkin technique. This is achieved by decomposing the spatial variation using basis functions. This is similar to decomposing a vector along some basis. The basis functions used in this study are not the eigenmodes of the linearised system, but just a set of functions which satisfy the boundary conditions. The ODEs obtained using the above technique are in the time domain.

These evolution equations are solved numerically using the 4th order Runge-Kutta scheme. This system of nonlinear ODEs is similar to the numerous dynamical systems in literature. The complete evolution equations are linearized and are found to be non-normal. It must be emphasized that the eigenvalues of the linearized equations are not the wave numbers of the basis functions used in the Galerkin technique.

3. A toy model for Rijke tube

A horizontal Rijke tube with an electric heat source is a system convenient for studying the fundamental principles of thermoacoustic instabilities both experimentally and theoretically. In such a set-up, the mean flow is provided by a blower, which sucks air in the tube. This enables us to control the heater power and the mean flow independently, and side-step modeling the effects of natural convection. If the tube were oriented vertically, as in the classical Rijke tube, the effect of mean flow component caused by natural convection will have to be accounted for in the stability analysis. The horizontal orientation of the Rijke tube is implemented to exclude the influence of natural convention on the mean flow rate. Such a setup has been used by Matveev [9-11], Heckl [6], and Kopitz and Polifke [34]. A schematic of the horizontal Rijke tube setup is shown in figure (3). Balasubramanian and Sujith [13] demonstrated the consequences of non-normality using a model for such a set-up.

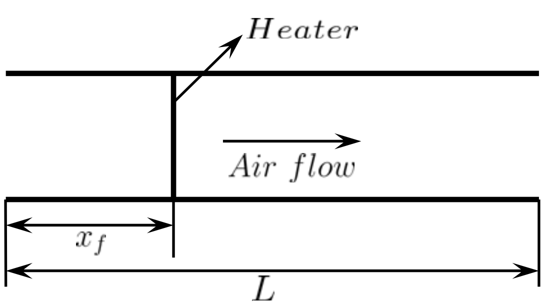

Figure 3: Schematic of a horizontal Rijke tube set-up.

3.1. Governing Equations

Assuming a perfect, inviscid and non-heat-conducting gas, the governing equations for the one-dimensional acoustic field in the absence of mean flow and mean temperature gradient in the duct, the governing equations for the one-dimensional acoustic field are [25]:

Acoustic Momentum:

$$\bar{\rho}\frac{\partial \tilde{u}'}{\partial \tilde{t}} + \frac{\partial \tilde{p}'}{\partial \tilde{x}} = 0 \tag{4}$$

Acoustic Energy:

$$\frac{\partial \tilde{p}'}{\partial \tilde{t}} + \gamma \bar{p}\frac{\partial \tilde{u}'}{\partial \tilde{x}} = (\gamma - 1)\dot{\tilde{Q}}' \tag{5}$$

where, $\tilde{p}$ is the acoustic pressure, $\tilde{u}$ is the acoustic velocity, $\dot{\tilde{Q}}'$ the unsteady heat release rate, and γ is the ratio of specific heats. A modified form of King's law is used to model the heat release rate. Since King's law exhibits nonlinearity only for velocity perturbations greater than the mean fluctuations, Heckl [6] suggested the following empirical model for the heat release rate:

$$\dot{\tilde{Q}}' = \frac{2L_w(T_w - \bar{T})}{S\sqrt{3}}\sqrt{\pi\lambda C_v \bar{\rho}\frac{d_w}{2}}\left[\sqrt{\left|\frac{u_0}{3} + u_f'(t-\tau)\right|} - \sqrt{\frac{u_0}{3}}\right]\delta(\tilde{x} - \tilde{x}_f) \tag{6}$$

In the above expression, L_w is the equivalent length of the wire, λ is the heat conductivity of air, C_v is the specific heat of air at constant volume, τ is the time lag, $\bar{\rho}$ is the mean density of air, d_w is the diameter of the wire, $(T_w - \bar{T})$ is the temperature difference, and S is the cross-sectional area of the duct.

A more detailed model will require modeling of the hydrodynamic zone, instead of using a correlation for the wire heat transfer. This increases the complexity of the problem, but brings much more richness to the problem [14, 15, 35]. The formalism for developing such a model is given in Section 7.3.

The above equations can be non-dimensionalized as follows;

$$\tilde{x} = L_a x;\ \tilde{t} = \frac{L_a}{c_0}t;\ \tilde{u}' = u_0 u';\ \tilde{p}' = \bar{p}p';\ M = \frac{u_0}{c_0}; \tag{7}$$

where c_0 is the speed of sound, L_a is the duct length and $\bar{p}$ is the pressure of the undisturbed medium and u_0 is the mean flow velocity. The acoustic equations in the non-dimensional form can be written as follows:

$$\gamma M\frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0 \tag{8}$$

$$\frac{\partial p'}{\partial t} + \gamma M\frac{\partial u'}{\partial x} = k\left[\sqrt{\left|\frac{1}{3} + u'(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\delta(x - x_f) \tag{9}$$

$$\text{where,}\quad k = (\gamma - 1)\frac{2L_w(T_w - \bar{T})}{Sc_0\bar{p}\sqrt{3}}\sqrt{\pi\lambda C_v\bar{\rho}\frac{d_w}{2}u_0} \tag{10}$$

The above set of partial differential equation can be reduced to ordinary differential equations using the Galerkin technique [35]. The velocity and pressure field can be written in terms of the duct's natural modes in the absence of the oscillatory heat release rate as given below [36, 37]:

$$u' = \sum_{j=1}^{\infty} \eta_j \cos(j\pi x) \quad \text{and} \quad p' = -\sum_{j=1}^{\infty} \frac{\gamma M}{j\pi} \dot{\eta}_j \sin(j\pi x) \qquad (11)$$

The Galerkin technique makes use of the fact that any function in a domain can be expressed as a superposition of expansion functions which form a complete basis in that domain. The basis functions are chosen such that they satisfy the boundary conditions. However the choice of the basis functions is not unique. The basis functions chosen here are just an arbitrary basis and are not the eigenfunctions of the system. They are the eigenfunctions of the self adjoint part of the linearized system. Clearly, the expansion functions chosen here satisfy the boundary conditions and they form a complete basis.

Substituting the above expansions into Eq. (8) and (9), and projecting along the basis functions the following evolution equations are obtained,

$$\frac{d\eta_j}{dt} = \dot{\eta}_j \qquad (12)$$

$$\frac{d\dot{\eta}_j}{dt} + k_j^2 \eta_j = -\frac{2k}{\gamma M} j\pi \left[\sqrt{\left| \frac{1}{3} + u'_f(t-\tau) \right|} - \sqrt{\frac{1}{3}} \right] \sin(j\pi x_f) \qquad (13)$$

In the presence of damping the above set of equations can be modified as follows [10]:

$$\frac{d\eta_j}{dt} = \dot{\eta}_j \qquad (14)$$

$$\frac{d\dot{\eta}_j}{dt} + 2\xi_j \omega_j \dot{\eta}_j + k_j^2 \eta_j = -\frac{2k}{\gamma M} j\pi \left[\sqrt{\frac{1}{3} + u'_f(t-\tau)} - \sqrt{\frac{1}{3}} \right] \sin(j\pi x_f) \qquad (15)$$

where the damping constant is given by:

$$\xi_j = \frac{1}{2\pi} \left[c_1\, \omega_j / \omega_1 + c_2 \sqrt{\omega_1 / \omega_j} \right] \qquad (16)$$

In this paper, the effect of damping is studied by varying the constants c_1 and c_2 in the above expression. In experiments, the changes in these constants in Eq. (16) can be effected by adjusting the end conditions and other experimental conditions. Equations (14) & (15) can be expanded to 2[nd] order for low amplitudes to yield the following matrix differential equation,

$$\frac{d\chi}{dt} + B_{NL}(\chi^T, \chi) + B_{NN}\chi = 0 \qquad (17)$$

where, $\chi = [\eta_1 \quad \dot{\eta}_1 \quad \eta_2 \quad \dot{\eta}_2 \; \ldots\ldots\ldots\ldots \eta_N \quad \dot{\eta}_N]^T$

$$B_{NN} = \begin{bmatrix} 0 & -1 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ \omega_1^2 + \beta_1 \cos(\pi x_f) & 2\omega_1\xi_1 & \beta_1 \cos(2\pi x_f) & 0 & \ldots & \ldots & \beta_1 \cos(N\pi x_f) & 0 \\ 0 & 0 & 0 & -1 & \ldots & \ldots & 0 & 0 \\ \beta_2 \cos(\pi x_f) & 0 & \omega_2^2 + \beta_2 \cos(2\pi x_f) & 2\omega_2\xi_2 & \ldots & \ldots & \beta_2 \cos(N\pi x_f) & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & \ldots & \ldots & 0 & -1 \\ \beta_N \cos(\pi x_f) & 0 & \beta_N \cos(2\pi x_f) & 0 & \ldots & \ldots & \omega_N^2 + \beta_N \cos(N\pi x_f) & 2\omega_N\xi_N \end{bmatrix} \qquad (18)$$

$$B_{NL} = u'(t-\tau) \begin{bmatrix} 0 & 0 & 0 & 0 & .. & 0 & 0 \\ \alpha_1 \cos(\pi x_f) & -\alpha_1\tau\cos(\pi x_f) & \alpha_1 \cos(2\pi x_f) & -\alpha_1\tau\cos(2\pi x_f) & .. & \alpha_1 \cos(N\pi x_f) & -\alpha_1\tau\cos(2\pi x_f) \\ 0 & 0 & 0 & 0 & .. & 0 & 0 \\ \alpha_2 \cos(\pi x_f) & -\alpha_2\tau\cos(\pi x_f) & \alpha_2 \cos(2\pi x_f) & -\alpha_2\tau\cos(2\pi x_f) & .. & \alpha_2 \cos(N\pi x_f) & -\alpha_2\tau\cos(N\pi x_f) \\ \ldots & \ldots & \ldots & \ldots & .. & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & .. & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & .. & 0 & 0 \\ \alpha_N \cos(\pi x_f) & -\alpha_N\tau\cos(\pi x_f) & \alpha_N \cos(2\pi x_f) & -\alpha_N\tau\cos(2\pi x_f) & .. & \alpha_N \cos(N\pi x_f) & -\alpha_N\tau\cos(N\pi x_f) \end{bmatrix} \qquad (19)$$

where, $\beta_j = \left(k / \gamma M\right) \sqrt{3}\, j\pi \sin(j\pi x_f),\ \alpha_j = \beta_j / 2$.

We normalize the system such that the 2-norm of the state vector is proportional to the acoustic energy (see Section 3.6). To achieve this, we multiply the system by a diagonal weight matrix

W = diag [1 (1/π) 1 (1/2π) ….. 1 (1/$N\pi$)].

The modified state vector is $W\chi$ and modified linearized operator is $WB_{NN}W^{-1}$. Various cases highlighting the non-normal and nonlinear nature of thermoacoustic oscillations and their consequences are discussed using examples in the following section.

3.2. Results and discussions

The role played by non-normality on thermoacoustic interaction can be studied through the following examples. The numerical simulations were performed by keeping some of the parameters fixed and varying others. The parameters that were kept fixed are γ = 1.4, $\lambda = 0.0328\ W/mK$, $C_v = 719\ J/kg\,K$, T =295 K, d_w = 0.0005 m and $\bar{\rho} =\ 1.205\ kg/m^3$. The set of first order ordinary differential equations (14)-(15) was integrated numerically using the 4[th] order Runge Kutta technique. The numerical simulations were performed with ten acoustic modes so that the change in the solution with increase in number of modes is less than 5%.

3.3. Triggering

If the system is non-normal as well as nonlinear, oscillations can grow even when the individual eigenvalues indicate linear stability. For such systems, there exists some initial condition for which the oscillations decay and some other initial conditions for which they grow. This feature is captured by a heater located at $x_f = 0.7$, $\bar{u} = 0.3\ m/s$ and $c_0 = 344.64\ m/s$, $L_w = 2.5\ m$, $T_w = 1000\ K$, $c_1 = 0.0415$, $c_2 = 0.0045$, $S_c = 0.0016\ m^2$. The time lag was chosen as $\tau = 0.01$. Figures (4a) and (4b) show triggering in the absence of damping. Figure (4a) shows that for an initial condition of $\eta_3(0) = 0.1\ \&\ \eta_5(0) = 0.1$ and $\eta_{i\neq1}(0) = 0$, the oscillations decay, whereas figure (4b) shows that the oscillations grow for a different initial condition; i.e., for $\eta_3(0) = 0.12\ \&\ \eta_5(0) = 0.1$ and $\eta_{i\neq1}(0) = 0$. Further, figure (4b) shows that the amplitude of the oscillations saturates. Figure (4c) shows the evolution of the phase between the acoustic oscillations and the heat release which finally saturates as the limit cycle is reached. The evolution of the non-dimensional acoustic energy ($\frac{1}{2}p'^2+\frac{1}{2}\left(\gamma M u'\right)^2$) is obtained after smoothing with moving average for both the linear (integration of Eq. 14 and 15 without nonlinear terms) and nonlinear simulations (with nonlinear terms included) and are presented in Figure 4(d). The linear simulation shows that the acoustic energy grows initially and eventually decays. The nonlinear simulation is almost identical to the linear simulation initially. After sufficient transient growth, the nonlinearity 'picks up' which can be seen from the deviation of the nonlinear evolution from the linear evolution.

The phenomenon of triggering has been observed in experiments with Rijke tube [10], and also in other thermoacoustic devices such as solid rocket motors [38, 39]. This is usually attributed to nonlinearities. However, as shown in this example, this is not the complete picture. An initial condition with very small initial amplitude, if applied in the optimal manner, can cause growth in the energy of the system. If this transient growth due to non-normality is high enough to make the nonlinear effects significant, the system which is stable according to classical linear stability theory can become nonlinearly unstable. Therefore for non-normal systems, a small amplitude initial perturbation which can be thought to be in the linear range can cause the nonlinear evolution to reach self-sustaining oscillations.

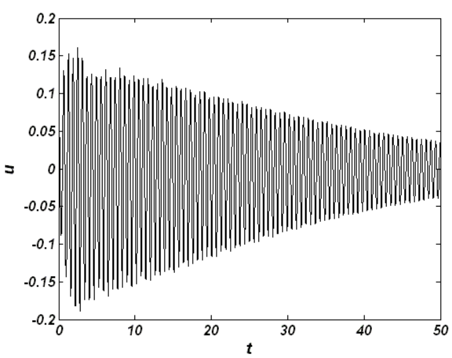

Figure 4a

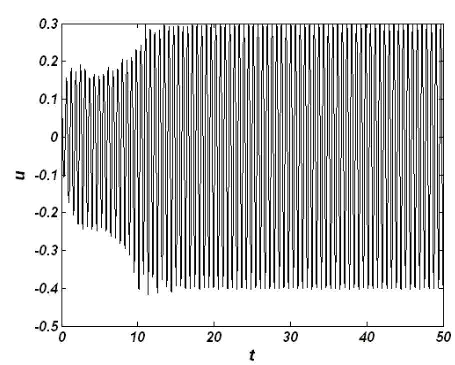

Figure 4b

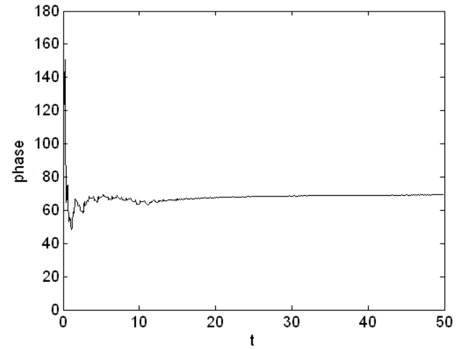

Figure 4c

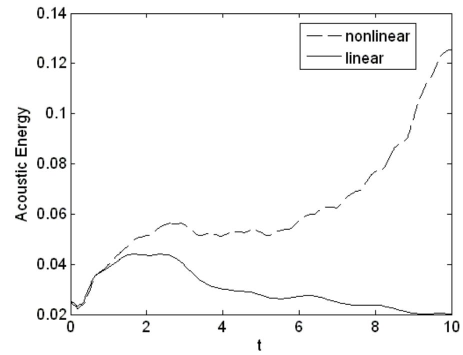

Figure 4d

Figure 4: shows the evolution of non-dimensional acoustic velocity when the initial condition is a) $\eta_3(0)=0.1$ & $\eta_5(0)=0.1$, $\eta_{i\neq1}(0)=0$; b) $\eta_3(0)=0.12$ & $\eta_5(0)=0.1$, $\eta_{i\neq1}(0)=0$. with the other parameters being $x_f=0.29$, $c_0=399.6\ m/s$, $L_w=3.6\ m$, $\bar{u}=0.5\ m/s$, $\bar{\rho}=1.205\ kg/m^3$, $T_w=1000\ K$, $C_v=719\ J/kgK$, $\lambda=0.0328\ W/mK$. c) Evolution of phase and d) linear and nonlinear evolution of the acoustic energy.

3.4. Growth of oscillations in an initially decaying system

This section discusses instability in a thermoacoustic system which is stable according to classical linear stability analysis based on eigenvalues and is initially decaying. In this example, the feature is captured by a heater located at $x_f=0.8$, $\bar{u}=0.3\ m/s$, $c_1=0.04$, $c_2=0.004$, $c_0=344.64\ m/s$. The initial conditions chosen are $\eta_1(0)=0.38$, $\eta_{i\neq1}(0)=0$ and $\dot{\eta}_i(0)=0$. L_a, L_w and T_w were chosen as 1 *m*, 2.5 *m* and 1680 *K*, respectively. The wire diameter was chosen to be 0.5 *mm*, the duct cross sectional area *S* was chosen to 0.00156 m^2 and time lag was chosen as $\tau=0.02$. Figure 5a shows the evolution of acoustic velocity at the heater location. It can be seen from the spectra of the evolution for the first quarter (25 non-dimensional time) of the evolution (Figure 5b) and that of the total (100 non-dimensional time) evolution (Figure 5c) that low frequency oscillations that are initially present in the system decays and high frequency oscillations sets in after some time (very close to the frequency of the 4th Galerkin mode). Further, it can be seen that the oscillations eventually saturate after nonlinear growth. It must be emphasized that classical linear stability analysis based on the eigenvalues shows all eigenmodes of this coupled system to be stable.

Figures 5d and e show the evolution of the acoustic velocity projected on the first and fourth Galerkin modes. It can be seen that while the acoustic velocity projected to the first Galerkin mode decays, the projection on the fourth mode grows. After sufficient energy is projected onto the 4th Galerkin mode, it projects back energy on the 1st Galerkin mode, also causing it to grow and then finally saturate along with it. We observe a shift in frequency during the evolution. The net effect of all these energy transfer causes the acoustic velocity to grow and eventually saturate. This feature has been discussed in the context of turbulence and it is known as "bootstrapping" [28, 32]. Yoon, et al. [8] has discussed "bootstrapping" in the context of Rijke tube using an ad hoc nonlinear model for the heat release rate.

There are two mechanisms by which energy can get redistributed in the system. The first mechanism is a nonlinear mechanism where two individual eigenmodes interact directly causing exchange of energy between these two modes. Another mechanism which causes redistribution of energy is the interaction of various modes with the base flow. At low amplitudes when the nonlinear effects are not significant, the redistribution of energy mainly occurs due to the interaction of various modes with the base flow. When the disturbance is caused by exciting only one eigenmode, it is expected that the oscillations will decay if the system is linearly stable. However, if the system is non-normal, then the oscillations can grow if there is a small amount of energy in the unexcited modes. This could be due to noise in the system or due to a mild nonlinearity.

The authors would like to emphasize that the mild nonlinearity just transfers some energy from the excited modes to other modes. Though, the individual modes can decay there can be an overall growth of amplitude due to non-normality. This overall growth can cause nonlinear effects to become significant and more energy can get exchanged between the various modes. Such a situation cannot occur in a linearly stable normal system (classical linear stability) if the nonlinearity is initially mild. This is because the amplitudes of a normal system decay if the individual eigenmodes decay and hence nonlinearity becomes milder and milder. This explains how non-normality plays an important role in the exchange of energy between various eigenmodes.

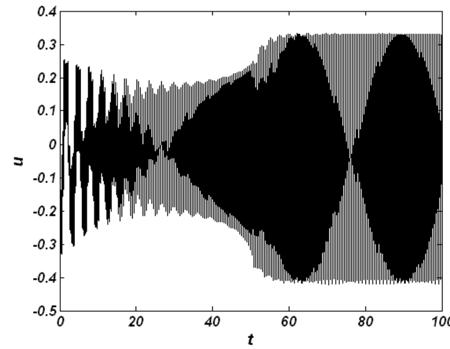

Figure 5a

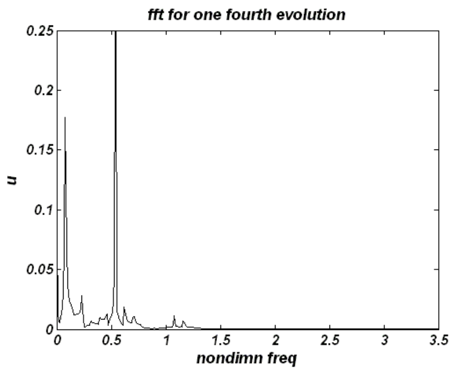

Figure 5b

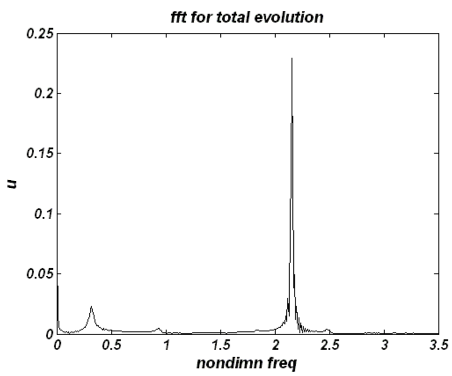

Figure 5c

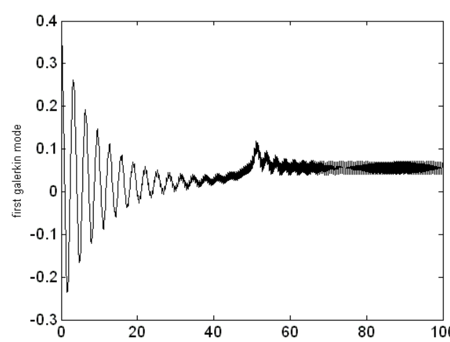

Figure 5d

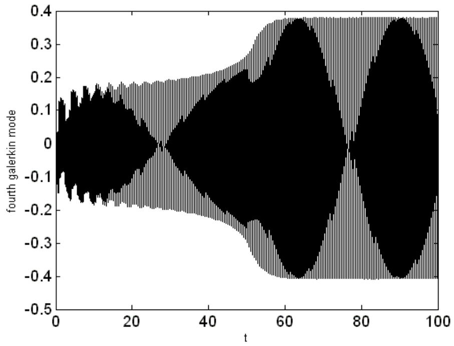

Figure 5e

Figure 5: (a) shows the non-dimensional evolution of acoustic velocity (b, c) the FFTs (d) projection on first Galerkin mode and (e) projection on 4th Galerkin mode

3.5. Transient growth

As discussed earlier, the non-normal nature of a system can cause transient growth of oscillations which can trigger nonlinearities in the system. In this section, a method to analyze transient growth is discussed. Schmid and Henningson [33] gave a detailed discussion on the analysis of transient growth, in the context of transition to turbulence in shear flows. They analyzed the stability of shear flows by studying the energy growth of the system. This analysis is general and can be applied to thermoacoustic systems as well. Balasubramanian and Sujith [12, 13] performed such an analysis in the context of thermoacoustic systems. The evolution of the acoustic oscillations in a Rijke tube is also non-normal as $LL^{\dagger} \neq L^{\dagger}L$. Here, L is the stability (or evolution) operator (The linearized dynamical system is represented by $d\chi/dt = L\chi$) The symbol $\dagger$ denotes the adjoint (conjugate transpose) of an operator. The solution of the linearized system of evolution equations [Eq. (14 & 15) without the nonlinear term] can be written in the operator form as [33] (Schmid & Henningson 2001):

$$\chi(t) = \exp(Lt)\chi(0) \qquad (20)$$

Transient growth is quantified by maximum growth factor which is defined as [33],

$$G(t) = \max_{\chi_0}\left(\|\chi(t)\|^2 / \|\chi(0)\|^2\right) = \|\exp(Lt)\|^2 \qquad (21)$$

where, 'max' indicates that the ratio of the norms is maximized over all initial conditions. Growth factor is a measure of maximum amplification of energy density at an instant of time. The expression in (21) is maximized for various instants of times, over all possible initial conditions. The maximum growth factor and the optimum initial condition were computed using singular value decomposition. The stability of a system can be studied using the maximum growth factor in a particular time interval [0, t] which is defined as $G_{\max} = \max_{t\geq 0} G(t)$. This maximum is obtained after smoothening $G(t)$.

The above maximum value is infinite if L has an eigenvalue with a positive real part. This corresponds to a linearly unstable system. In the present paper, $G_{\max}$ values for various parameters such as time lag and heater location are calculated and the regions with large transient growth are identified. Transient growth cannot occur for those parameters which have $G_{\max} = 1$. When $G_{\max} = 1$, the energy at any instant is less than the initial energy of the system indicating that the energy of the system decays. When $G_{\max} > 1$, the system will exhibit transient growth. Hence a system that follows the linear evolution initially and has G_{max} =1 will follow the linear evolution for all time, as there is no transient growth to trigger nonlinear effects. This fact is used in the next section to obtain necessary and sufficiency conditions for the stability of the system.

In the results obtained from this toy model, the growth factor is in the range of 2 to 5. The authors would like to point out that in the present analysis a simple model for the heating element is used. This simplified approach was used to focus on the non-normal nature of the acoustic equations alone in the presence of heat source. However, the energy released at the heating element is governed by an advection diffusion equation. It has been shown that the advection-diffusion equation is non-normal [26, 27]. This will cause the growth factor to be much larger as the number of eigenmodes of the coupled system will be much larger as can be seen from the calculations of Mariappan et al. [14].

3.6. Transient Energy Growth: Interpretation

In this section, we demonstrate the procedure to compute the transient energy growth and its physical interpretation in the context of thermoacoustic systems [38]. Norm plays a central role in computations of the transient energy growth. The norm of a mathematical object is a quantity that in a physical or an abstract sense describes the length, size, or extent of the object [39]. The 2-norm of a vector is its length in the Euclidean space. Thus on $\Re^p$ or a p-dimensional vector space over the Eucledian space $\Re$, a vector denoted by $x = (x_1, x_2, ..., x_p)$ has its 2-norm defined as $\sqrt{x_1^2 + x_2^2 + ... + x_p^2}$. $\chi(t)$ is a 2N dimensional vector space at a time t given by Eq. (7). The square of the norm $\|\chi(t)\|$ can thus be represented as follows:

$$\|\chi(t)\|^2 = \sum_{i=1}^{N}\left(\eta_i^2 + \frac{\dot{\eta}_i^2}{k_i^2}\right) \qquad (22)$$

The non-dimensional acoustic energy at any time t is given by the following equation:

$$E(t)=\int_0^1\left(\frac{1}{2}p'(x,t)^2+\frac{1}{2}(\gamma Mu'(x,t))^2\right)dx \quad (23)$$

On substituting the Galerkin expansions from Eq. (4) in Eq. (11), it can be easily shown that [40].

$$E(t)=\left(\frac{\gamma M}{2}\right)^2\|\chi(t)\|^2 \quad (24)$$

Hence the greatest possible energy growth at time *t*, maximized over all possible initial perturbations $\chi(0)$ is measured by the growth function defined as follows [24, 33]:

$$G(t)=\max_{\chi(0)}\frac{\|\chi(t)\|^2}{\|\chi(0)\|^2}=\|\exp(-Lt)\|^2 \quad (25)$$

$\|\exp(-Lt)\|$ is the 2-norm of the matrix exponential $\exp(-Lt)$. For any general matrix *A*, the 2-norm can be geometrically interpreted as the radius of the smallest circle that contains the image of a unit disk under the map *A*. Mathematically, the 2-norm of any matrix is its principal singularvalue [39]. It can be computed by singularvalue decomposition (SVD), using standard subroutines available in most software libraries. SVD of a general $n\times m$ matrix *A* is defined as $A=U\Sigma V^T$. *U* is a $n\times n$ unitary matrix, Σ is a $n\times m$ matrix with non-negative numbers on its diagonal and zeros off the diagonal and V^T is the transpose of *V*, which is *a* $m\times m$ unitary matrix. Further, the diagonal of Σ consists of $\min(n,m)$ elements; arranged in a descending order. These diagonal elements are known as the singularvalues of A. Geometrically, *A* can be visualized as an operator that maps a sphere of unit vectors in vector space S_1 into an ellipsoid in vector space S_2 with the singularvalues as the lengths of the principle radii of the ellipsoid. The directions of the principle radii are the column vectors of *U* and the directions of their pre-images are the columns of *V*.

In light of the above discussion, Eq. (20) can be rewritten as follows:

$$\chi(t)=U\Sigma V^T\chi(0) \quad (26)$$

Equation (14) can be interpreted from an input-output view point. $V^T\chi(0)$ resolves the initial condition vector $\chi(0)$ into an orthonormal basis of input vectors v_i (i^{th} column *V*). Each of the components are amplified by the corresponding singularvalues σ_i (diagonal element of Σ). $U\Sigma V^T\chi(0)$ represents the output vector $\chi(t)$ as a linear superposition of components along the orthonormal basis formed by the output vectors u_i (i^{th} column of *U*). For *i* = 1, we can write the following equation:

$$e^{-Lt}v_1=\sigma_1 u_1 \quad (27)$$

Since SVD returns the singularvalues in a descending order, v_1 is the most sensitive (highest gain) input direction and u_1 is the most sensitive (highest gain) output direction. Thus from Eq. (27), σ_1 signifies the maximum possible gain. Hence, the principle singularvalue of a matrix gives the 2-norm or the maximum amplification of the energy. The corresponding right singularvector gives the most sensitive or the optimal initial condition for maximum energy growth. It must be emphasized that the maximum amplification obtained by the above procedure is for a particular time instant *t* maximized over all initial conditions (Eq. (13)). Thus, in a given time interval [0, *t*], the global maximum amplification factor is given as follows:

$$G_{\max}=\max_t\left(G(t)\right)=\max_t\left(\|\exp(-Lt)\|^2\right) \quad (28)$$

Similar geometric interpretation of the optimal initial condition required for maximum growth is provided by Farrell and Ioannou [40] in the context of atmospheric flows. The maximum growth rate that can be attained at time *t* is greater or equal to that of the fastest growing eigenvector at *t* and the minimum growth rate (or the least singularvalue) is less than or equal to that of the slowest growing eigenvector [40]. This can be translated to the following inequality:

$$\sigma_{\min}\le e^{(\lambda_i+\lambda_i^*)t}\le G(t)=\sigma_{\max}=\sigma_1 \quad (29)$$

where λ_i is the eigenvalue of $(-L)$, λ_i^* is conjugate of λ_i, $\sigma_{\min}$ is the least singularvalue of $\exp(-Lt)$ and σ_1 is the principal singularvalue of $\exp(-Lt)$.

Unlike SVD, Eigenvalue Decomposition (EVD) does not provide an orthogonal input and output vector basis for systems far from normality. EVD is not an appropriate tool for obtaining transient growth and the most optimal initial condition direction for the maximum energy amplification. The eigenvalues obtained by EVD determine only the asymptotic behavior of any linear dynamical system. In normal systems, as there is no possibility of transient growth, the stability predicted by the eigenvalues remains valid even at finite time. However, if the system is far from normal, no concrete conclusions about its transient behavior can be made based on eigenvalues alone. Thus, EVD is not sufficient to study the complete dynamics of a non-normal thermoacoustic system.

SVD provides a powerful method to analyze the transient behavior of non-normal systems. As seen in this section, the maximum energy growth can be determined from the singularvalues. Using this, in the next section, we characterize the maximum energy growth for the Rijke tube model.

3.7. Pseudospectra

The degree of resonant amplification that may occur in a normal system in response to an input frequency is inversely proportional to the distance in the complex plane between the input frequency and nearest eigenvalues. However, in the case of a non-normal operator, the resonant amplifications may vary non-uniformly with the proximity to eigenvalues. The amplification can be large, even when the input frequency is far from any eigenvalue [28, 41].

The concept of ε–pseudoeigenvalues can be used to analyze the behavior of evolution governed by such non-normal operators [27, 28, 41]. z is an ε pseudoeigenvalue of A if it satisfies $\left\|(zI - A)^{-1}\right\| \geq \varepsilon^{-1}$. There are other equivalent definitions of pseudoeigenvalues and they have been discussed in great detail by Trefthen and Embree [27, 28, 41]. It should be noted that the systems discussed in this paper is a self-excited system. In a linearly stable self-excited system, if the initial condition is along one of the eigenmodes, then there is no growth and the oscillations will decay throughout the evolution. However, in most situations, the initial conditions do not correspond to any particular eigenmode and hence transient growth can occur. Transient growth and the non-normal nature of the operator can be studied using pseudospectra.

The pseudospectra of normal operators are closed circles. For a non-normal operator, when a contour corresponding to some ε value spills to the right half of the spectrum more than the perturbations in the eigenvalues, the system can exhibit transient growth causing the amplitudes to increase to high values [28].

Trefethen *et al.* [28, 41] have used the relation between the geometry of the pseudospectra and the lower bound on the transient growth factor to analyze hydrodynamic instability in Couette and Poiseuille flows. The lower bound on the transient growth factor is defined in terms of the Kreiss constant $\kappa(A)$ as given below [43].

$$\sup_{t\geq 0}\left\|e^{tA}\right\| \geq \sup_{\varepsilon>0}\frac{\alpha_{\varepsilon}(A)}{\varepsilon} \equiv \kappa(A) \tag{30}$$

Here, $\alpha_{\varepsilon}(A)$ is the pseudospectral abscissa and it gives the abscissa of a point on the ε-pseudospectra with the largest real part. The above inequality suggests that if the pseudospectra of A protrude into the right half plane such that $\alpha_{\varepsilon}(A) > \varepsilon$, then the system can exhibit transient growth.

Rayleigh criterion [2] gives the condition for acoustic driving to occur. However, the prediction of transient growth by Rayleigh criterion requires the precise knowledge of initial conditions. The ambiguity of initial conditions due to noise makes the identification of transient growth using Rayleigh criterion difficult. However, the condition obtained in terms of pseudospectra are conditions on the evolution operator and hence do not depend on the initial conditions [12, 13].

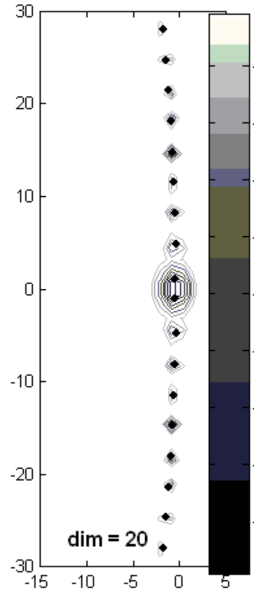

Figure 6: Pseudospectra for a Rijke tube with parameters $x_f = 0.95$, $\bar{u} = 0.1\ m/s$ and $c_0 = 344.64\ m/s$, L_w =2m and T_w = 1300 K, S = 0.0016 m^2, $\tau = 0.01$, c_1 = 0.415 and c_2 = 0.0045.

Figure (6) shows the pseudospectra for a Rijke tube with parameters $x_f = 0.95$, $\bar{u} = 0.1\ m/s$ and c_0 = 344.64 m/s, L_w =2 m and T_w = 1300 K, S = 0.0016 m^2, $\tau = 0.01$, c_1 = 0.415 and c_2 = 0.0045. It is clear from the non-circular behavior that the system is non-normal. It is clear that the contour is not entirely on the left half plane. Hence, it can be inferred from Eq. (30) that this system shows transient growth. This indicates that there is an "unstable" pseudo-eigenvalue for some ε. Even if a system behaves linearly, the transient growth can cause the amplitudes to reach high values and trigger nonlinearities which can cause the oscillation to grow further.

4. Non-normality in ducted premixed flame-acoustic interaction

Stringent emission requirements drive operating conditions of premixed gas turbines and combustors to the lean regime. However, lean premixed combustion has been shown to be particularly susceptible to combustion instability [42, 43]. Subramanian and Sujith [16] investigated the role of non-normality and nonlinearity in flame-acoustic interaction in a ducted premixed flame.

The premixed flame thermoacoustic system is modeled by considering the acoustic momentum and energy equations together with the equation for the evolution of the flame front obtained from the kinematic G-equation [44]. The G-equation is rewritten as the front-tracking equation as given in Eq. (32) for an axi-symmetric wedge flame in a purely axial flow [45] shown in figure 7. The scales used for the non-dimensionalisation are derived from the length of the flame $(b/sin\alpha)$ and velocity of the flow $\bar{u}$ as shown below in equation (32).

$$X = \tilde{X}\sin\alpha/b;\ \bar{u} = \tilde{\bar{u}}/\tilde{\bar{u}} = 1; t = \tilde{t}/(b/\bar{u}\sin\alpha) \quad (31)$$

$$\frac{\partial\xi'}{\partial t} = (1+u')\cos\alpha\left(\frac{\partial\xi'}{\partial X}\right) - (1+u')\sin\alpha - \sqrt{1+\left(\frac{\partial\xi'}{\partial X}\right)^2} \quad (32)$$

The configuration of a duct which is open at both ends with the axi-symmetric wedge flame stabilized at an axial location within it is considered.

The linear operator of a generic thermoacoustic system has been shown to be non-normal [12, 13]. The linearised operator for the premixed flame thermoacoustic system is also found to be non-normal leading to non-orthogonality of its eigenvectors. Non-orthogonal eigenvectors can cause transient growth in evolutions even when all the eigenvectors are decaying for a linearly stable case. Therefore, classical linear stability theory cannot predict the finite time transient growth observed in non-normal systems. A parametric study of the variation in transient growth due to change in parameters such as flame location and flame angle is performed. It is found that the transient growth is pronounced when the flame has a small angle of flame and is located close to the center of the duct. The optimum initial condition which causes maximum transient growth can be identified using singular value decomposition (SVD) for a given system configuration. In addition to projections along the acoustic variables of velocity and pressure, the optimal initial condition for the self evolving system has significant projections along the variables for heat release rate.

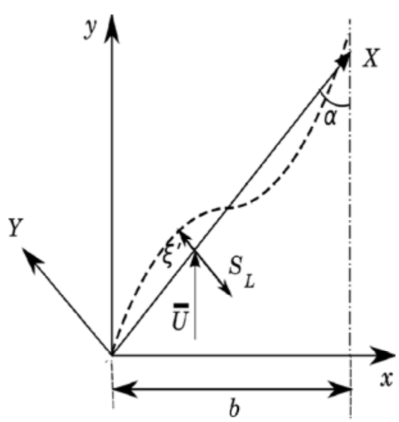

Figure 7. Geometry of an axi-symmetric wedge flame stabilized on a wire. Here ξ' is the displacement of instantaneous flame shape from the unperturbed flame shape along X, α is the flame angle, S_L is the laminar flame speed, b is the radius of the burner and $\bar{u}$ is the mean flow.

Nonlinear simulations show subcritical transition to instability from a small but finite amplitude perturbation to the system as shown in figure 8. The system is perturbed with the optimal initial perturbation with initial acoustic velocity of small but finite amplitude $u'_f(0) = 0.07$ at the flame location. It shows that the linear and nonlinear evolutions diverge within a short period of time. Asymptotically, the linear evolution decays as shown in figure 8(*a*) while the nonlinear evolution reaches a self sustained oscillation of amplitude $|u'_f| = 0.89$ as seen in the inset from figure 8(*b*). Therefore an initial condition with very small initial amplitude, if applied in the optimal manner, can cause transient growth in the energy of the system.

Therefore, the premixed flame thermoacoustic system has more degrees of freedom than the number of acoustic modes. These additional degrees of freedom represent the internal degrees of freedom of the flame front or the internal flame dynamics. These internal degrees of the flame front must be preserved in the thermoacoustic model to accurately capture the non-normal effects. In thermoacoustic systems, subcritical transition to instability has been thought of as being caused by a large amplitude initial perturbation to a linearly stable system. In a linearly stable case, even a small but finite amplitude optimal initial perturbation is shown to reach limit cycle. The optimum initial condition may have contributions from variables that represent the flame front dynamics, and will not be purely acoustic. Therefore in non-normal systems, initial perturbations with amplitudes that are small when compared with the limit cycle oscillations can cause subcritical transition to instability.

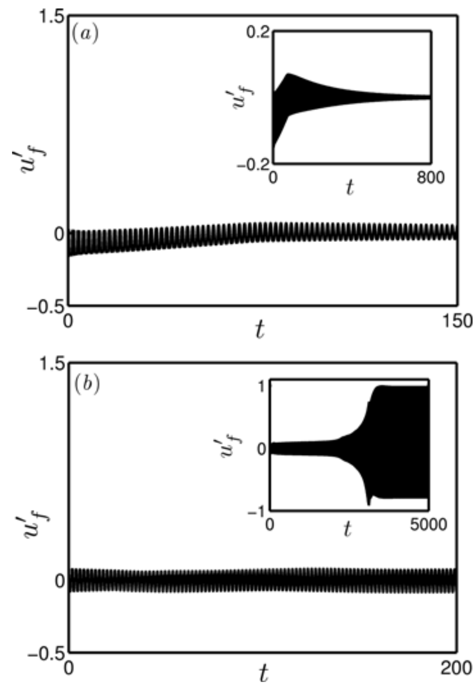

Figure 8: Evolution of acoustic velocity at the flame for (*a*) linearised system and (*b*) nonlinear system with $\alpha = 10^0$, $y_f = 0.8$, $\phi = 0.6$, $c_1 = 4\times10^{-3}$, $c_2 = 4\times10^{-4}$, $S_L = 0.1231$ *m/s* and $\triangle q_R = 1.6885\times10^6$ *J/Kg*. Initial condition is $u'_f(0) = 0.07$.

5. Non-normality in ducted diffusion flame-acoustic interaction

Balasubramanian and Sujith [12] investigated the role of non-normality and nonlinearity in flame-acoustic interaction in a ducted diffusion flame. They used the infinite rate chemistry model to study unsteady diffusion flames in a Burke-Schumann type geometry. The combustion response to perturbations of velocity is non-normal and nonlinear. This flame model is then coupled with a linear model of the duct acoustic field to study the temporal evolution of acoustic perturbations. The one-dimensional acoustic field is simulated in the time domain using the Galerkin technique, treating the fluctuating heat release from the combustion zone as a compact acoustic source. The coupled combustion-acoustic system is non-normal and nonlinear. An analysis of transient growth revealed that the growth factor increased monotonically with the ratio of the acoustic to combustion length scale and Peclet number. However, the variation of growth factor with the nondimensional slot width was not monotonic.

The model exhibited the occurrence of triggering; i.e., the thermoacoustic oscillations decay for some initial conditions whereas they grow for some other initial conditions. The role of non-normality and nonlinearity in triggering instabilities starting from small but finite initial conditions is brought out. The occurrence of "pseudoresonance" occurring at frequencies far from the system's natural frequencies is highlighted. Non-normal systems can be studied using pseudospectra, as eigenvalues alone are not sufficient to predict the behavior of the system. Further, both necessary and sufficient conditions for the stability of a thermoacoustic system are presented in this paper.

6. Thermoacoustic instability in solid rocket motor: non-normality and nonlinear instabilities

Solid rocket motors (SRMs) are often prone to combustion instability. The prediction of combustion instability in the early design stage is a formidable task due to the complex unsteady flow field existing in the combustion chamber. Combustion instability occurs when the unsteady burn rate from the propellant (in SRMs) is amplified by the positive feed back of the acoustic oscillations in the chamber. Combustion instability causes excessive pressure oscillations, which might resonate with the structural modes of the rocket, leading to excessive vibration and damage of the payload. Further, during the occurrence of combustion instability, the heat transfer to the combustion chamber walls is increased, eventually melting them [48]. Instabilities in SRMs are known to exist since 1930 [49]. Since then, many investigations were conducted to understand the mechanisms behind them and arrive at measures to control the same. Mariappan and Sujith [20] investigated the role of non-normality in the occurrence of triggering instability in solid rocket motors with homogeneous propellant grain.

Theoretical analysis starts with linearising the governing equations and analyzing their stability. This leads to finding the eigenvalues (complex frequency) and eigenmodes of the system. In classical linear stability analysis, a system is said to be linearly stable if the oscillations decay to zero in the asymptotic time limit, reaching finally the steady state (stable fixed point). The system is linearly unstable if the oscillations grow exponentially. Both the definitions are for 'small' disturbances with respect to the corresponding mean quantities. Non-normal systems show initial transient growth for suitable initial perturbations even when the system is stable according to classical linear stability theory. Transient growth plays an important role in the subcritical transition to instability regime [20]. In SRM the above is termed as 'pulsed instability', where the system is linearly (small amplitude) stable, but nonlinearly (large amplitude) unstable. This section focuses on the role of transient growth during pulsed instability.

The SRM considered here has a prismatic cylindrical propellant grain of length 'l', port circumference 'S_l' and a constant port area 'S_c'. A schematic of the geometry considered with the coordinate system used is shown in figure 9(a). A cylindrical geometry is studied, so as to make the analysis simple. The acoustic momentum and energy equations are used to model the chamber acoustic field. Galerkin technique [36, 37] is used to solve the above equations. The equation for unsteady burn rate from the propellant, which drives the chamber acoustic field, is taken from Krier *et al.* [49]. The derivation of the equation is given in Krier *et al.* [49] and the final nonlinear equation is as follows.

At each 'x' location,

$$\frac{\partial T}{\partial \tau} - (1+R)\frac{\partial T}{\partial y} - \frac{\partial^2 T}{\partial y^2} = 0\,,\ 0 \le y < \infty,\ 0 \le \tau < \infty \quad (33)$$

$$R = T_S^{m_p} - 1$$

$$T_S(\tau) = T(y=0,\tau)$$

$$\tau/t = \left(l\bar{R}^2\right)\Big/\left(a\alpha\right) = F$$

Boundary Condition (BC):

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = -\frac{(1+p)^{2n}\left((1+p)^{n/m_p} - H\right)}{1+R} - H(1+R) \quad (34)$$

$$T(y \to \infty, \tau) = 0$$

Initial Condition (IC):

$$T(y,0) = T_{st}(y) + T_p^0(y) \quad (35)$$

where, $y = \tilde{y}\big/\left(\alpha/\bar{R}\right),\ \tau = t\left(l/a\right)\big/\left(\alpha/\bar{R}^2\right),$ $H = Q_S\Big/\left(S_p\left(\tilde{T}_{S,0} - \tilde{T}_\infty\right)\right),\ T = \left(\tilde{T} - \tilde{T}_\infty\right)\Big/\left(\tilde{T}_{S,0} - \tilde{T}_\infty\right),$

$T_{st} = e^{-y}$, $\tilde{T}_\infty$ is the temperature of the propellant at $y \to \infty$, $\tilde{T}_{S,0}$ is the surface temperature of the propellant, T_{st} is the non-dimensional steady state temperature, T_p^0 is the non-dimensional temperature fluctuation at t = 0, n is the burn rate index, m_p is the pyrolysis coefficient, α is the thermal diffusivity of the propellant, Q_S is the overall heat release per unit mass at the propellant surface, S_p is the specific heat capacity of the propellant, y is the non-dimensional distance from the propellant surface, H is the ratio of the heat release at the propellant surface to its thermal capacity and F is the ratio of timescales of the chamber acoustics (τ_a) and transient heat conduction in the propellant (τ_{th}). The co-ordinate system is fixed to the propellant surface which regresses according to the burn rate. The geometry is shown in figure 9(b). A Dirichlet type boundary condition far from the propellant surface and a Neumann type at the surface (2), which comes from the balance between the amount of heat transfer from the flame in the gas phase to the propellant surface, are applied.

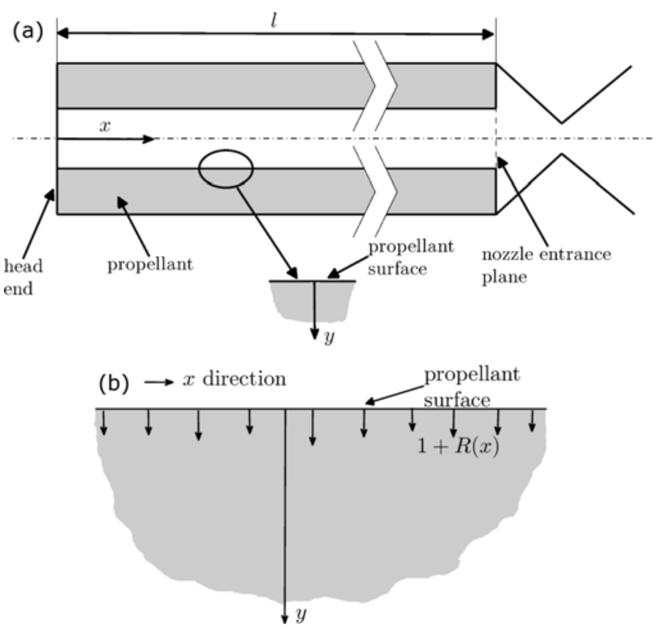

FIGURE 9. a) Schematic diagram of the combustion chamber geometry of the SRM considered, b) Geometry of the pressure coupled propellant response model; reproduced from reference [20], with permission from Cambridge University Press.

Pulsed instability can possibly occur in two ways. The first one is when the initial disturbance amplitude is large enough $\left(E_{ac}\left(t=0\right)=8.43\right)$ for the nonlinear terms to be dominant compared to the linear terms. Numerical simulations of the linearised equations shows decaying acoustic energy $\left(E_{ac}\left(t\right)\right)$ (figure 10a). This means that the system is linearly stable. Now, for the same parameters and initial condition, the nonlinear simulation shows that the acoustic energy initially decays and after sometime it starts growing with the amplitude eventually reaching a limit cycle.

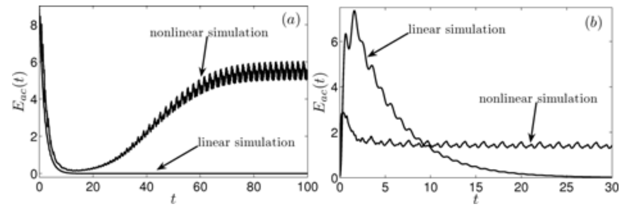

FIGURE 10. The evolution of acoustic energy $\left(E_{ac}\left(t\right)\right)$ from linear and nonlinear simulations, a) Large amplitude initial condition $\left(E_{ac}\left(t\right)\right)$ b) Small amplitude optimum initial condition; reproduced from reference [20], with permission from Cambridge University Press.

The second route is by non-normal transient growth. Here even if one starts with a finite small amplitude $\left(E_{ac}\left(t=0\right)=6.4\times10^{-4}\right)$ suitable initial condition, transient growth due to the non-normal nature of the system makes the oscillation grow even for a system stable according to linear stability theory. The transient growth leads to large amplitude oscillations, which causes the nonlinear terms to play dominant roles. Figure 10(b) shows the comparison of acoustic energy evolution with linear and nonlinear simulations. The transient growth in the linear simulation decays eventually, while the nonlinear simulation leads to a limit cycle. The higher transient growth in the linear simulation than in the nonlinear one is due to the damping effect from the nonlinear terms. Here, the initial condition is chosen to be the optimum initial condition for the maximum transient growth to show the importance of this route to triggering.

7. Summary and Conclusions

In this paper, we examine the nature of thermoacoustic interactions. The coupled thermoacoustic system is non-normal and hence the eigenvectors are non-orthogonal. Non-normality leads to short time amplification even though the individual modes decay exponentially. This transient growth can trigger nonlinearities when the amplitude of the fluctuations is sufficiently large.

In thermoacoustic systems, triggering (subcritical transition to instability) has been thought of as being caused by a large amplitude initial perturbation to a linearly stable system. However, for a linearly stable system, even a small but finite amplitude perturbation along the optimal direction can take the system to the basin of attraction of the limit cycle. Further, the optimal initial condition may have contributions from variables that represent the flame dynamics, and need not be purely acoustic in nature.

Singularvalue decomposition (SVD) is a convenient tool for the analysis of non-normal operators. The principle singularvalue obtained by SVD provides the maximum possible growth rate of a non-normal system in a given time interval. SVD also provides the optimal initial conditions for this maximum growth. The maximum possible energy growth is greater than or equal to the growth rate of the fastest growing eigenmode in the same time and the minimum possible energy growth is less than or equal to the growth rate of the slowest growing eigenmode of the system.

Transient growth in a non-normal system greatly depends on the system parameters which characterize the maximum growth factor (G_{max}) and numerical abscissa of the system. A system is only capable of exhibiting transient growth if the numerical abscissa is greater than zero [38]. The maximum growth factor gives the ratio of the magnitude of the maximum transient growth possible to the initial magnitude supplied [38]. However, apart from all these characterizations, whether a system finally exhibits transient growth depends on the initial conditions supplied. A linear system with a very high G_{max} and positive numerical abscissa will not have any transient growth if the initial condition is along the direction of one of the eigenvectors. In order to achieve the maximum transient growth, the optimal initial condition should be in the direction of the bi-orthogonal (or covariant) of the slowest decaying eigenmode [40]. Thus we see that the nature of evolution for a non-normal system depends highly on the type of initial condition (both direction and magnitude). For such systems sometimes the background noise which can be mathematically thought as an initial condition that could cause transient growth, and hence cause further instability.

For a non-normal system, "pseudoresonance" is shown to occur at frequencies far from the spectrum. The stability and sensitivity of non-normal systems can be studied using pseudospectra. The pseudospectra of normal operators are disjoint circles. The pseudospectra of the thermoacoustic system are non-circular implying a highly non-normal nature of the system. If the pseudospectra are not entirely within the left-half plane then the system can show transient growth. It is possible to identify systems which cannot show transient growth by analyzing the pseudospectra. The geometry of the pseudospectra provides necessary and sufficient conditions for the stability of a system.

The current methodology to study the onset of thermoacoustic oscillations involves looking for exponentially growing or decaying modes by calculating the individual eigenvalues of the linearized system. Further, the nonlinear behavior of the combustion response is modeled using flame transfer functions which are amplitude dependent. These approaches fail to predict phenomena such as nonlinear growth triggered by the transient growth which results from the nonnormality of the thermoacoustic system and excitation of frequencies that are not initially excited. More sophisticated approaches have to be taken and more involved methods have to be introduced, to accurately capture the transient behaviour which is critical for the overall system stability. However, the current linear system identification tools in time domain, along with the tools based on the linearized operator suggested in this paper such as growth factor and pseudospectra, can indeed be used to predict transient growth [50].

8. Outlook for the future

8.1. Characterizing bifurcations in thermoacoustic systems

The nonlinear dynamical behavior of a thermoacoustic system is best characterized by a bifurcation diagram. Numerical continuation is used to compute the numerical solutions to a set of parameterized nonlinear equations where other approaches to solve the problem are prohibitively expensive. But more importantly it can be used as a tool to gain insight into the qualitative properties of the solutions. It is used to calculate the bifurcations or qualitative changes in the solution for the variation of one or more parameters of the system. Solutions which are connected to a given state of the system are computed. Bifurcations are identified by including multiple test functions which change sign at the critical value of the parameter.

This method has the advantage that once a stationary or periodic solution has been computed, the dependence of the solution on the variation of a parameter is obtained very efficiently. It can also be used to compute unstable limit cycles. The restrictions on this approach are that the different types of equations, encountered in models of physical systems, require special attention during the stages of analysis and the reduced order modeling. Presently at IIT Madras, this technique has been applied to thermoacoustic systems such as a nonlinear Rijke tube model [51] and to the ducted premixed flame model (Unpublished) to compute bifurcations of the system behavior.

8.2. Adjoint Looping to determine the optimum initial condition

Juniper [52] developed a procedure based on the adjoint looping of the nonlinear governing equations to find the lowest initial energy and the corresponding initial state that can trigger self-sustained oscillations. He applied this to the low order Rijke tube model developed by Balasubramanian and Sujith [13]. Continuation method used in conjunction with adjoint looping may provide both physical insight and engineering solution to studying subcritical bifurcations in thermoacoustic systems.

8.3. Asymptotic Analysis for Coupling CFD and Acoustic Solvers

In recent years Computational Fluid Mechanics (CFD) and Computational Acoustics (CA) have evolved into mature subjects. Their role in the investigation of combustion instabilities needs to be examined in the context of their application to practical systems. CFD and CA has been used in recent years to study the combustion instability. While finite element modeling has been successfully used in modeling the complex acoustic field in complicated geometries [53], CFD has been successfully used to understand the unsteady combustion process [54].

The most complete solution to the problem is to solve conservation equations for mass, momentum and energy subject to appropriate boundary conditions. Numerical simulation has been performed in recent years, but it is hampered by the variety of scales. Furthermore, the use of CFD to do the full compressible flow simulations is prohibitively expensive. Therefore, a commonly adopted methodology is to determine the combustion response using a CFD analysis and then using this response as an input to the acoustic model (e.g., references [55-57]). All models with the exception of a few [58, 59] have used a linear response function. Even the studies that used nonlinear response have used a transfer function of the form $\frac{\hat{Q}/\overline{Q}}{\hat{u}/\overline{u}} = T_{Q,u}(\omega, A_u)$. Such a model can capture saturation in heat release. However, the equation used to study the combustion response are ad hoc; i.e., they are not derived based on a rigorous and mathematically consistent separation of the acoustic and combustion/hydrodynamic zones.

The typical length of the acoustic resonating duct (l_a) is of the order of metres and the dimension of the heat source along the axial direction of the duct (thickness, l_c) is of the order of millimetre and the Mach number (M) is of the order 10^{-3}. The thickness of the heat source is very small compared to the length scale of the acoustic length scale. Hence the heat source can be assumed to be compact compared to the acoustic length scale of the tube. The zone around the heat source is termed as the hydrodynamic zone. The length of the hydrodynamic zone in the axial direction of the duct is also of the order of the thickness of the heat source. Hence the hydrodynamic zone can also be assumed to be compact compared to the acoustic field (acoustic zone).

Thermoacoustic instability analysis is the study of the dynamics of the coupled system, comprising of the acoustic field in the tube and the unsteady heat transfer from the heat source (hydrodynamic zone). Therefore, it is important to track variations on the length scale of the acoustic zone and on the length scale of the radius of the hydrodynamic zone. Further, the acoustic time scale $t_{ac} = l_a/c$ (c-speed of sound) and the time scale $t_{cc} = l_c/u$ (u-velocity of the base flow) in the hydrodynamic zone are of the same order for typical values mentioned above. This leads to an effective coupling of the dynamics of the acoustic field and the unsteady heat release rate from the heat source. The problem has two length scales separated by a large factor ($l_c/l_a \rightarrow 0$) and one time scale ($t_{ac}/t_{cc} \sim 1$). Since the flow is at very low Mach number ($M \sim 10^{-3}$), the fluid dynamics equations become ill-conditioned [60]. Moreover, a smaller grid size near the heater will reduce the maximum timestep that can be achieved for the numerical scheme. All these make the governing equations stiff. As a consequence, solving the problem using CFD is a difficult task.

A standard technique available to solve this kind of two scale problem is asymptotic analysis [61]. Asymptotic analysis performed in the limit of compact heat source and zero Mach number of the steady flow leads to two systems of equations: one governing the acoustic field and the other governing the unsteady flow and heat transfer near the heat source [62]. The separation of equations for the acoustic field and heat source occurs in an elegant way. The coupling between the above two systems of equation appears naturally. Further, a non-trivial additional term appears in the momentum equation of the hydrodynamic zone, which has serious consequences on the stability of the system. This additional term is the global acceleration term. The presence of pseudo-acceleration term is generic, when there are two length scales involved in the problem.

The size of the linearized operator in such a system is much higher than what can be performed using SVD. Therefore, Mariappan et al. [14] used the technique of adjoint optimization to arrive at the optimum initial condition for a model thermoacoustic problem - Rijke tube, with a detailed model for the dynamics of the heat source.

8.4. Experimental efforts

The ideas of non-normality and transient growth are being pursued by the fluid dynamics community for nearly two decades, resulting in a large body of theoretical and numerical studies in the literature. However, there are just a handful of experimental results. The lack of experimental studies in the author's opinion is due to the inherent difficulty in extracting the relevant details (say for example growth factor) from experiments.

So far, there have been no published experimental studies on non-normality and transient growth in thermoacoustic oscillations. The study of nonlinear effects has been mostly confined to determining the nonlinear heat release response of flames performed in experiments where the flame is forced by externally imposed velocity fluctuations. Not many studies carefully examine the nonlinear effects, in particular subcritical transitions (triggering instabilities) in thermoacoustic systems. The author hopes that in the near future there will be serious effort to study both non-normality and nonlinearity in self-excited thermoacoustic systems.

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